Degenerate quasilinear Schrödinger equations

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- Local and global existence
- Wavefront behavior
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Dispersive Equations

- Equations where different wavenumbers propagate with different group velocities
- Plane wave solutions $Ae^{i(kx-\omega t)}$ give frequency in terms of wavenumber: $\omega = \omega(k)$ (dispersion relation)
- Phase velocity $\frac{\omega}{k}$ and group velocity $\frac{d\omega}{dk}$ differ
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Asymptotic equation for unidirectional shallow water waves:

The KdV equation

\[ u_t + u_{xxx} + 6uu_x = 0 \]

- Completely integrable
- Existence of soliton solutions
- *Soliton*: A localized wave of constant speed that emerges from collisions with other solitons unchanged but for a phase shift.

KdV soliton

\[ \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x - ct - a) \right) \]
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Craig, Kappeler, and Strauss considered

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\begin{align*}
  u_t + f(u_{xxx}, u_{xx}, u_x, u, x, t) &= 0
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and showed well-posedness as long as \( \partial_{u_{xx}} f \leq 0 \) and \( \partial_{u_{xxx}} f \geq \epsilon > 0 \).

- \( \partial_{u_{xx}} f \leq 0 \) prevents behavior associated with backwards heat equation.
- Uniform bound on \( \partial_{u_{xxx}} f \) ensures dispersive effects dominate, leading to smoothing.
- What if there’s no such uniform bound on \( \partial_{u_{xxx}} f \)?
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Rosenau and Hyman proposed:

The $K(m, n)$ equations

$$u_t + (u^m)_x + (u^n)_{xxx} = 0$$

- When $u \to 0$, dispersive effects due to $u_{xxx}$ vanish.
- This makes a local existence result harder (can’t use local smoothing due to dispersion).
- Numerical evidence indicates that $K(2, 2)$ is ill-posed in $H^2$ (Ambrose-Simpson-Wright-Yang).
- Why $H^2$?
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Compactons

Searching for travelling wave solutions to $K(2, 2)$ yields

\[ u(x, t) = \frac{4c}{3} \cos^2 \left( \frac{x - ct}{4} \right) \chi[-2\pi, 2\pi](x - ct) \]

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When can we get local existence results for degenerate dispersive equations?

**Theorem (Ambrose and Wright)**

For $k \in \pm 1$, $L_0 > 0$, and $\phi \in H^2([-L_0, L_0])$, there exists $T^*(L_0, k, \phi) > 0$ such that for all $0 < T < T^*$ there is a weak solution $u \in L^2(0, T, H^{7/4})$ of the Cauchy problem for the equation

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Another example of nonlinear dispersion:

**Semilinear Schrödinger Equation**

\[ iu_t + \frac{1}{2} \Delta u + \mu |u|^{\alpha-1} u = 0 \]

- \( u \) is complex-valued, in contrast to the KdV-type equations
- \( \mu \in \{-1, 1\} \) (focusing versus defocusing), \( \alpha > 1 \) (most commonly \( \alpha = 3 \))
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A generalization

If we instead consider

**Semilinear Schrödinger II**

\[ u_t = i\Delta u + F(u, \bar{u}, \nabla \bar{u}) \]

then standard energy estimates will yield local well-posedness for \( s > n/2 + 1 \) under modest assumptions on the boundedness of \( F \) and its derivatives.
Why didn’t $F$ depend on $\nabla u$?

- Unless the coefficient of $\nabla u$ is real, it can’t be eliminated in a straightforward manner during energy estimates.
- If $v = \partial^{\alpha} u$ then estimating $\partial_t |v|^2$ requires dealing with terms like $\bar{v}\nabla v - v\nabla \bar{v}$, so integration by parts won’t work.
- Alternatively, we can think of $u_t = iu_x$ (for $u \in \mathbb{C}$) as a complex realization of the Cauchy-Riemann equations, which are ill-posed as evolution equations.
- Hayashi-Ozawa: deal with this by making a substitution and obtaining estimates for an equivalent energy.
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A simple example

Consider the case $n = 1$ and the equation

**Toy NLS**

$$u_t = iu_{xx} + uu_x + u\bar{u}_x$$

Taking three derivatives, we linearize in $v = \partial_x^3 u$

$$v_t = iv_{xx} + uv_x + u\bar{v}_x + l.o.t.$$ 

Multiply this equation by $e^\phi$ and rearrange, letting $w = e^\phi v$

$$w_t = iw_{xx} + (u - 2i\phi_x) e^\phi v_x + ue^\phi \bar{v}_x + l.o.t.$$
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Choice of substitution

To get a system in \( w \) that we can perform estimates on we should choose

\[
\phi(x, t) = -\frac{i}{2} \int_0^x u(x', t) dx'
\]

The equation for \( w \) now has no \( w_x \) terms, so the standard estimates will work.

- In fact, we only need to eliminate the imaginary part of the \( u_x \) term: if \( b(x) \) is our \( u_x \) term then

Integrability Condition (Hayashi-Ozawa 1994)

\[
\sup_{x \in \mathbb{R}} \left| \int_0^x \text{Im} b(t) dt \right| < \infty
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will be sufficient for \( L^2 \)-well-posedness.
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The general quasilinear framework

Kenig, Ponce, and Vega considered the equation

Quasilinear Schrödinger

\[ u_t = ia_{jk}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u})u_{x_j x_k} \]
\[ + \vec{b}_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla u + \vec{b}_2(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla \bar{u} \]
\[ + c_1(x, t, u, \bar{u})u + c_2(x, t, u, \bar{u})\bar{u} + f(x, t) \]

\( H^s \) local well-posedness was established under lots of assumptions.
- Ellipticity of \( a_{jk} \)
- Regularity and decay at infinity of coefficients and their derivatives
- The Hamiltonian flow associated to the symbol of the initial data is nontrapping
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Comments on the KPV result

- Nontrapping condition: implies an analogous integrability condition for variable coefficient operators
- Proof uses artificial viscosity method: insert $-\epsilon \Delta^2 u$ term to regularize the problem
- Pseudodifferential operators give a local smoothing estimate on linearized solutions, allowing an $\epsilon$-independent existence time
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A degenerate quasilinear Schrödinger equation

DNLS Equation

\[ iA_t + (|A|^2 A_x)_x = 0 \]

- Arises as an asymptotic equation for a two-wave system in compressible gas dynamics
- Quantum-mechanical interpretation: a free particle with mass inversely proportional to probability density.
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Two natural generalizations

First, we generalize to arbitrary functions of $|A|^2$:

**DNLS II**

$$iA_t + (\rho(|A|^2)A_x)_x = 0$$

We can also consider the equation in more than one spatial dimension:

**DNLS: $n$ dimensions**

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Hamiltonian formulation

The original equation $iA_t + (|A|^2 A_x)_x = 0$ can be written in the Hamiltonian form

$$\mathcal{H}(A, A^*) = \frac{1}{4} i \int AA^*(AA^*_x - A^* A_x) dx$$

$$A_t + \partial_x \left[ \frac{\delta \mathcal{H}}{\delta A^*} \right] = 0$$

Conserved quantities in addition to $\mathcal{H}$ include the action and the momentum:

$$S = \frac{1}{2} i \int (A \partial_x^{-1} A^* - A^* \partial_x^{-1} A) dx$$

$$\mathcal{P} = \frac{1}{2} \int AA^* dx$$
Hamiltonian formulation

The original equation $iA_t + (|A|^2 A_x)_x = 0$ can be written in the Hamiltonian form

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Hamiltonian in $n$ dimensions

It appears that the $n$-dimensional equation cannot be put in Hamiltonian form for $n > 1$. A modified equation that is Hamiltonian:

$$iA_t + \nabla \cdot (|A|^2 \nabla A) - A|\nabla A|^2 = 0$$

The Hamiltonian form of this equation is

$$\mathcal{H}(A, A^*) = \int AA^* \nabla A \cdot \nabla A^* \, dx$$

$$iA_t = \frac{\delta \mathcal{H}}{\delta A^*}$$

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**Modified Hamiltonian DNLS**

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Approach

Goal: Obtain local existence of solutions to the DNLS equation.

Two steps:

1. Get local existence for the equation

\[ iA_t + (\rho(|A|^2)A_x)_x = 0 \]

in the case that \( \rho \) is bounded away from zero.

2. See if it’s possible to get an existence time independent of the bound on \( \rho \) (or find some way to pass to a limit in a suitable sense)
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Proposition (Local Existence)

Suppose $A_0 \in H^3(\mathbb{T})$, $\rho \in C^4(\mathbb{R})$ and there exist $\rho_0$, $C > 0$ such that $\rho_0 \leq \rho(\alpha) \leq C|\alpha|$. Then there exists $T > 0$ depending on $\rho$, $\rho_0$, $C$ and $A \in L^2(0, T; H^3(\mathbb{T})) \cap C(0, T; H^2(\mathbb{T}))$ satisfying

\[
\left\{ \begin{array}{c}
  iA_t + (\rho(|A|^2)A_x)_x = 0 \\
  A(x, 0) = A_0(x)
\end{array} \right.
\]
Proof Sketch

- Use parabolic regularization to obtain solutions on a time interval of $O(\epsilon)$
- $L^2$ estimate on $\rho^{7/4}A_{xxx}$ gives a uniform existence time and allows us to pass to a limit
- The assumption that $\rho$ is bounded away from zero is critical: otherwise this estimate is not equivalent to an estimate on $\|A_{xxx}\|_{L^2}$. 
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In the degenerate case we have $\rho(|A|^2) = |A|^2$, so our estimate is

$$\int |A|^7 |A_{xxx}|^2 \lesssim 1$$

However, this does not translate to an $H^3$ estimate.

We can better understand behavior of solutions to the degenerate equation by considering some numerical simulations (courtesy of John Hunter).
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Numerical Simulations
Numerical observations

The numerics demonstrate a few important properties of solutions:

- Finite speed of propagation for compactly supported data
- A waiting time prior to dispersing outward from initial data
- Well-behaved modulus with rapid oscillations near the boundary

These observations lead us to consider local behavior near the wavefront of a solution. Two methods for doing this:

1. Whitham’s averaged Lagrangian method
2. Similarity solutions (motivated by study of porous medium equation)
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The degenerate Schrödinger equation has Lagrangian

$$S(v, v^*) = \int -\frac{1}{2}(v^*_x v_t + v_x v^*_t) + \frac{1}{4} iv_x v^*_x (v^*_x v_{xx} - v_x v^*_x) dx dt$$

where $v = \partial_x^{-1} A$.

Because of this, we can apply the averaged Lagrangian method of Whitham:

- Write $v = \rho(x, t)e^{\frac{i\phi(x,t)}{\epsilon}}$, substitute into Lagrangian
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Letting $k = \phi_x$, $\omega = -\phi_t$, our averaged Lagrangian is

$$\bar{S}(\rho, \phi) = \int k\omega \rho^2 - \frac{1}{2}k^5 \rho^4 \, dx \, dt$$

$\bar{S}$ should solve the Euler-Lagrange equations

$$\frac{\delta \bar{S}}{\delta \rho} = \frac{\delta \bar{S}}{\delta \phi} = 0$$
Averaged Lagrangian

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Results of Euler-Lagrange equations

Adding in the compatibility condition $k_t + \omega_x = 0$ yields the 2-D system:

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\begin{align*}
  k_t + (k^4 \rho^2)_x & = 0 \\
  (k \rho^2)_t + \frac{3}{2} (k^4 \rho^4)_x & = 0
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\]

- This system is hyperbolic in $k$ and $\eta = k \rho^2$ (solve by Riemann invariants)
- What happens as $|A| \to 0$? Analysis suggests $\rho k \frac{1 - \sqrt{3}}{2}$ approaches a time-independent state.
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Porous Medium Equation

Finite propagation speed of solutions, waiting time prior to moving wavefront invites comparison to PME and other nonlinear diffusion equations. For instance:

\[ u_t = \nabla \cdot (u^n \nabla u) \]

- Multiple families of similarity solutions
- Existence of a family of waiting-time solutions (comes from similarity solutions in one space variable)

Idea: derive analogous similarity solutions
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Similarity solutions to DNLS

- Search for similarity solutions of the form $A(x, t) = t^{-1/4} f(\xi)$ for $\xi = \frac{x}{t^{1/4}}$ and $f \in \mathbb{C}$. This yields the ODE

$$-\frac{i}{4} \xi f + |f|^2 f' = C$$

- If we choose $C = 0$ we obtain the one-parameter family of non-degenerate solutions

$$A(x, t) = A_0 t^{-1/4} \exp \left( \frac{ix^2}{8\sqrt{t}|A_0|} \right)$$

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One way DNLS arises is as an asymptotic equation for the MRS equation. (Hunter 1995)

\[ \begin{align*}
    u_t + uu_x &= v \\
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Deriving DNLS from MRS

- Multiple scale expansion: let \( \tau = \epsilon^2 t \) and write
  \[
  u \sim \epsilon u_1(x, t; \tau) + \epsilon^2 u_2(x, t; \tau) + \ldots \\
  v \sim \epsilon v_1(x, t; \tau) + \epsilon^2 v_2(x, t; \tau) + \ldots
  \]

- At \( O(\epsilon) \), we get the linearized solution:
  \[
  u = A(x, \tau)e^{it} + \text{c.c.} \\
  v = iA(x, \tau)e^{it} + \text{c.c.}
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- At \( O(\epsilon^3) \) we get the solvability condition
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Returning to MRS

- We want to investigate the lifespan of small solutions to the MRS equation.
- Notice: solutions to the linearized equation oscillate with constant frequency in time and do not enjoy any dispersive decay.
- This property is similar to the inviscid Burgers-Hilbert equation.
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Inviscid Burgers-Hilbert

\[ u_t + uu_x = H[u] \]

Suppose our initial data is small, say \[ ||u_0||_{H^s} \leq \epsilon \]. What sort of lifespan can we expect for a solution?

- Standard energy estimates yield a quadratic lifespan
- Can we expect to do any better than this?
Degenerate Schrödinger Equations
MRS equation
Moving forward

Lifespan for Burgers-Hilbert

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$$u \mapsto u + B(u, u)$$

where $B(u, u)$ is quadratic.

**Normal form transformation: inviscid Burgers-Hilbert**

$$u \mapsto u + \mathbf{H}[\mathbf{H}u \cdot \mathbf{H}u_x]$$

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**Solution:** Define a modified energy $E_k$. What do we want from $E_k$?

1. $E_k(u) \sim \| \partial_x^k u \|_{L^2}^2$
2. $\frac{dE_k}{dt}$ contains no quadratic or cubic terms.

Modified energy: inviscid Burgers-Hilbert

$$E_k = \frac{1}{2} \| \partial_x^k u \|_{L^2}^2 + \langle \partial_x^k u, \partial_x^k H[H u \cdot H u_x] \rangle$$

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- Yields cubic lifespan for small solutions
The modified energy method

**Solution:** Define a modified energy $E_k$. What do we want from $E_k$?

1. $E_k(u) \sim \|\partial_x^k u\|_{L^2}^2$
2. $\frac{dE_k}{dt}$ contains no quadratic or cubic terms.

**Modified energy: inviscid Burgers-Hilbert**

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- Yields cubic lifespan for small solutions
What about MRS?

- Wagenmaker (1994) showed that solutions have a logarithmically improved lifespan.

- Normal form transformation for MRS:

\[
\begin{align*}
  u &\mapsto u + \frac{1}{3} \partial_x \left( u^2 - uv + \frac{1}{2} v^2 \right) \\
  v &\mapsto v - \frac{1}{3} \partial_x \left( \frac{1}{2} u^2 - uv + v^2 \right)
\end{align*}
\]

- This transformation suffers the same loss of derivatives as in Burgers-Hilbert.
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Modified Energy for MRS

As in Burgers-Hilbert, throw out highest-order terms from normal form transformation.

\[
E_k = \frac{1}{2} ||\partial_x^k u||^2_{L^2} + \frac{1}{2} ||\partial_x^k v||^2_{L^2} \\
+ \frac{1}{3} \langle \partial_x^k u, \partial_x^{k+1} (u^2 - uv + \frac{1}{2} v^2) \rangle - \frac{1}{3} \langle \partial_x^k v, \partial_x^{k+1} (\frac{1}{2} u^2 - uv + v^2) \rangle
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Unfortunately, \( E_k \) is not equivalent to \( ||\partial_x^k u||^2 + ||\partial_x^k v||^2 \), so the method fails.
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MRS modified energy

\[
E_k = \frac{1}{2} \| \partial_x^k u \|^2_{L^2} + \frac{1}{2} \| \partial_x^k v \|^2_{L^2} + \frac{1}{3} \langle \partial_x^k u, \partial_x^{k+1}(u^2 - uv + \frac{1}{2}v^2) \rangle - \frac{1}{3} \langle \partial_x^k v, \partial_x^{k+1}(\frac{1}{2}u^2 - uv + v^2) \rangle
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Unfortunately, \( E_k \) is not equivalent to \( \| \partial_x^k u \|^2 + \| \partial_x^k v \|^2 \), so the method fails.
As in Burgers-Hilbert, throw out highest-order terms from normal form transformation.

\[ E_k = \frac{1}{2} \left\| \partial_x^k u \right\|_{L^2}^2 + \frac{1}{2} \left\| \partial_x^k v \right\|_{L^2}^2 + \frac{1}{3} \left\langle \partial_x^k u, \partial_x^{k+1} \left( u^2 - uv + \frac{1}{2} v^2 \right) \right\rangle - \frac{1}{3} \left\langle \partial_x^k v, \partial_x^{k+1} \left( \frac{1}{2} u^2 - uv + v^2 \right) \right\rangle \]

Unfortunately, \( E_k \) is not equivalent to \( \left\| \partial_x^k u \right\|^2 + \left\| \partial_x^k v \right\|^2 \), so the method fails.
Future Research

- Long-term goal: local existence result for the DNLS equation
- Analyze similarity solutions
- Investigate wavefront behavior
- Study modified versions of DNLS and other related degenerate dispersive equations
- Understand normal form methods for these equations